p-adic Schwarzian triangle groups of Mumford type

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We define a certain p-adic analogue of classical Schwarzian triangle groups related to Mumford's uniformization of analytic curves and give a complete classification of it.

1. Introduction

1.1. *Uniformization of orbifolds and triangle groups.* The rich geometric structure of uniformized analytic varieties over non-archimedean fields has been studied by many authors, and already has a long history. Mumford [Mum72] showed that an analytic curve $\mathcal{X}$ defined over a non-archimedean field K with a split multiplicative analytic reduction can be uniformized as $\mathcal{X} \cong \Gamma\backslash(\mathbb{P}^{1,\mathrm{an}}_K - \mathcal{L}_\Gamma)$, where Γ is a finitely generated free discrete subgroup of $\mathrm{PGL}(2, K)$ and $\mathcal{L}_\Gamma$ is the set of limit points of Γ. An equally important example is the uniformization of a curve which is an étale covering of a Mumford curve, studied by van der Put [vdP83]. These are the most practical and reasonable analogues of uniformizations of complex analytic curves.

Historically, however, the theory of uniformization in complex analysis arose from interplay between geometric and function-theoretic viewpoints. This is apparent if one considers the orbifold uniformization of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}}$ with finitely many points marked by positive integers (= the ramification degrees). The link between geometry and function theory stems from the behavior of the multivalued function z, inverse to the uniformization map, which is written as a ratio $z = u_1/u_2$ of two linearly independent solutions of a Fuchsian differential equation with rational exponents (cf. [Yos87]). For example, if the orbifold is $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}}$ with precisely the three points $0, 1, \infty$ marked by $e_0, e_1, e_\infty \in \mathbb{Z}_{>0}$, then the corresponding differential equation is the Gaussian hypergeometric equation

$$x(1-x)\frac{d^2u}{dx^2} + \{c - (a+b+1)x\}\frac{du}{dx} - abu = 0 \qquad (1.1.1)$$

(where $a, b, c \in \mathbb{Q}$ with $|1-c| = 1/e_0$, $|c-a-b| = 1/e_1$, and $|a-b| = 1/e_\infty$). Depending on whether $1/e_0 + 1/e_1 + 1/e_\infty > 1$, $= 1$ or < 1, the image of $z = u_1/u_2$ is isomorphic to $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}$, respectively, and z maps the upper-half plane onto the interior of a triangle region with the angles π/e_0, π/e_1 and π/e_∞. The corresponding orbifold fundamental group has a representation into $\mathrm{PGL}(2, \mathbb{C})$ with discrete image, the so-called *Schwarzian triangle group* $\Delta(e_0, e_1, e_\infty)$, whose action on the universal covering space is visible in terms of complex reflections (cf. [Mag74, Chap. II]).

The problems which arises from carrying out such a program in the p-adic situation are mainly topological. It is perhaps appropriate here to remind the reader of

the fact that in rigid analysis an étale covering map is not necessarily a topological covering map (= the locally topologically trivial map). Consequently, in contrast to complex analysis, we have many simply connected domains, even one dimensional; for instance, the complement of finitely many points in $\mathbb{P}^{1,\mathrm{an}}_K$ is simply connected. In particular, a reasonable definition of the orbifold fundamental groups is highly non-trivial.

In [And98], Y. André studied the rather special class of étale covering maps which are composites of topological coverings followed by finite étale (not necessarily topological) coverings. He observed that the covering maps of this kind give rise to a reasonable concept of orbifold fundamental groups (denoted by π_1^{orb} in [And98]) in the p-adic situation, and discussed the relation with differential equations; one of his result phrases it as follows [And98, §6]: Consider the orbifold $\mathcal{X}$ (cf. [And98, 5.1] for the precise definition) which is supported on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ with n-points ζ_i marked by positive integers e_i ($1 \leq i \leq n$). Then:

There exists a canonical fully faithful functor of categories

$$\left\{\begin{array}{l}\textit{Continuous representations } \rho \\ \textit{of } \pi_1^{\mathrm{orb}}(\mathcal{X}, \overline{x}) \textit{ into } \mathrm{GL}(r, \mathbb{C}_p) \\ \textit{with discrete coimage}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\textit{Algebraic regular connections} \\ \textit{on } \mathbb{P}^1_{\mathbb{C}_p} - \{\zeta_i \,|\, 1 \leq i \leq n\} \\ \textit{of rank } r \textit{ such that the local} \\ \textit{monodromy at each } \zeta_i \textit{ is of fi-} \\ \textit{nite order dividing } e_i\end{array}\right\}. \tag{1.1.2}$$

Moreover, the essential image of this functor consists of the connections ∇ enjoying the following condition (Global Monodromy Condition):

(1.1.3) *There exists a connected rigid analytic curve $\mathcal{S}$ and a finite morphism $\varphi: \mathcal{S} \to \mathcal{X}$ ramified above precisely the points ζ_i with ramification indices dividing e_i, such that the connection $\varphi^*\nabla$ on $\mathcal{S}$ admits a full set of multivalued analytic solutions on $\mathcal{S}$ (and, moreover, on the Berkovich space associated to $\mathcal{S}$).*

In particular, if $n = 3$, $r = 2$ and if the image Γ of ρ in $\mathrm{PGL}(2, \mathbb{C}_p)$ is discrete, then Γ can be regarded as a p-adic analogue of the Schwarzian triangle group $\Delta(e_0, e_1, e_\infty)$. If so, Γ gives the "projective monodromy" for the connection ∇ defined by the functor (1.1.2), which is nothing but the one associated to the Gaussian hypergeometric equation (1.1.1). In [And98, §9], André discussed such groups, called *p-adic triangle groups*, and gave a complete list of the so-called *arithmetic* p-adic triangle groups, which are constructed through the Cherednik-Drinfeld theory of uniformization of Shimura curves, starting from Takeuchi's list of arithmetic triangle groups. Notably, he deduced that *there exists no arithmetic p-adic triangle groups for $p > 5$*. The uniformizations of the orbifolds $\mathcal{X}$ corresponding to these groups are given by the Drinfeld upper-half plane or its étale coverings.

1.2. *Results of this paper.* In this paper, we will discuss (not necessarily arithmetic) p-adic triangle groups Γ as above, especially in the case that the corresponding uniformization is given by the space $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} - \mathcal{L}_\Gamma$. We call such a group Γ a p-adic triangle group *of Mumford type*; we can define it in simpler terms (without involving π_1^{orb}) as follows:

Definition. A finitely generated discrete subgroup Γ of $\mathrm{PGL}(2,\mathbb{C}_p)$ is said to be a *p-adic (Schwarzian) triangle group of Mumford type* if $\Gamma\backslash(\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}-\mathcal{L}_\Gamma)\cong\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ and the uniformization map

$$\varpi_\Gamma\colon(\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}-\mathcal{L}_\Gamma)\longrightarrow\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$$

is ramified above precisely three points.

If Γ is a finite subgroup, then $\mathcal{L}_\Gamma=\emptyset$, and the map ϖ_Γ is the analytification of the algebraic quotient map $\mathbb{P}^1_{\mathbb{C}_p}\to\mathbb{P}^1_{\mathbb{C}_p}/\Gamma$. Hence the spherical (i.e., $1/e_0+1/e_1+1/e_\infty>1$) p-adic triangle groups of Mumford type amount to the same as the classical ones. Our main theorem gives the complete classification of the other p-adic triangle groups of Mumford type:

Theorem. (1) *A non-spherical p-adic triangle group Γ of Mumford type of index (e_0,e_1,e_∞) exists if and only if p and the unordered triple (e_0,e_1,e_∞) occur in the left two columns of Table 1, where, in the last row, the integers l,m,n obey the following condition;*

$(*)$ *$lmn\geq 4$, and at least two of l,m,n are odd.*

In each of the cases in Table 1, the conjugacy class of Γ in $\mathrm{PGL}(2,\mathbb{C}_p)$ is uniquely determined. In particular, p-adic triangle groups of Mumford type do not exist if $p>5$.

Table 1: List of p-adic triangle groups of Mumford type

	p	Index	
I	5	$(3,3,5l)$	$(l\geq 1)$
II	5	$(2,3,5l)$	$(l\geq 2)$
III	3	$(5,5,3l)$	$(l\geq 1)$
IV	3	$(2,5,3l)$	$(l\geq 2)$
V	3	$(4,4,3l)$	$(l\geq 1)$
VI	3	$(2,3l,3l)$	$(l\geq 2)$
VII	3	$(2,4,3l)$	$(l\geq 2)$
VIII	3	$(4,5,3l)$	$(l\geq 1)$
IX	2	$(5,5,2l)$	($l\geq 1$ and l: odd)
X	2	$(5,5,4l)$	$(l\geq 1)$
XI	2	$(5,2l,4m)$	($lm\geq 1$ and l: odd)
XII	2	$(3,3,2l)$	($l\geq 3$ and l: odd)
XIII	2	$(3,3,4l)$	$(l\geq 1)$
XIV	2	$(3,2l,4m)$	($lm\geq 2$ and l: odd)
XV	2	$(3,5,2l)$	($l\geq 3$ and l: odd)
XVI	2	$(3,5,4l)$	$(l\geq 1)$
XVII	2	$(2m+1,2l,2l)$	$(m\geq 1,l\geq 2)$
XVIII	2	$(2l,2m,2n)$	(l,m,n satisfy $(*)$)

(2) *In each of the cases in Table 1, Γ is isomorphic to the abstract group given as follows (here, Z_n denotes the cyclic group of order n, D_n the dihedral group of*

degree n ($\cong Z_n \rtimes Z_2$), etc. The symbol $*$ means the amalgam product, that is, the push-forward in the category of groups.):

$$
\begin{array}{lcl}
\mathrm{I} & : & A_5 *_{D_5} D_{5l} *_{D_5} A_5, \\
\mathrm{II} & : & A_5 *_{D_5} D_{5l}, \\
\mathrm{III} & : & A_5 *_{D_3} D_{3l} *_{D_3} A_5, \\
\mathrm{IV} & : & A_5 *_{D_3} D_{3l}, \\
\mathrm{V} & : & S_4 *_{D_3} D_{3l} *_{D_3} S_4, \\
\mathrm{VI} & : & A_4 *_{Z_3} Z_{3l}, \\
\mathrm{VII} & : & S_4 *_{D_3} D_{3l}, \\
\mathrm{VIII} & : & S_4 *_{D_3} D_{3l} *_{D_3} A_5, \\
\mathrm{IX} & : & \begin{array}{c} \qquad D_{2l} \\ \qquad *_{D_2} \\ A_5 *_{A_4} A_4 *_{A_4} A_5, \end{array}
\end{array}
\qquad
\begin{array}{lcl}
\mathrm{X} & : & A_5 *_{A_4} S_4 *_{D_4} D_{4l} *_{D_4} S_4 *_{A_4} A_5, \\
\mathrm{XI} & : & \begin{array}{c} \qquad A_5 \\ \qquad *_{A_4} \\ D_{4m} *_{D_4} S_4 *_{D_2} D_{2l}, \end{array} \\
\mathrm{XII} & : & A_4 *_{D_2} D_{2l}, \\
\mathrm{XIII} & : & S_4 *_{D_4} D_{4l} *_{D_4} S_4, \\
\mathrm{XIV} & : & D_{4m} *_{D_4} S_4 *_{D_2} D_{2l}, \\
\mathrm{XV} & : & A_5 *_{D_2} D_{2l}, \\
\mathrm{XVI} & : & A_5 *_{A_4} S_4 *_{D_4} D_{4l} *_{D_4} S_4, \\
\mathrm{XVII} & : & D_{2m+1} *_{Z_2} Z_{2l}, \\
\mathrm{XVIII} & : & D_{2l} *_{D_2} D_{2m} *_{D_2} D_{2n}.
\end{array}
$$

Here, in I, III, V, VIII, the two dihedral groups with the same order are chosen to be equal if l is odd, and are not equal and are conjugate with each other by an involution in the dihedral group between denoted between them if m is even. In X, XIII, XVI, and XVIII, the two dihedral groups with the same order are chosen to be equal, and in XI and XIV, the subgroups D_4 and D_2 in S_4 are chosen such that $D_2 \subset D_4$. In XI, the subgroup A_4 in S_4 is chosen so that its intersection with D_2 is trivial.

Remarks. (1) The theorem, in particular, proves Yves André's conjecture that there are infinitely many non-arithmetic p-adic triangle groups (cf. [And98]).

(2) In each of the cases in the table, at least one of the numbers e_i ($i = 0, 1, \infty$) is divisible by the residue characteristic p.

(3) The theorem shows that there are no Euclidean (i.e., $1/e_0 + 1/e_1 + 1/e_\infty = 1$) p-adic triangle groups of Mumford type. The reason for this is that the elliptic curve which covers an Euclidean orbifold always has a complex multiplcation and never be a Tate curve.

(4) The theorem and the well-known fact on automorphisms of Mumford curves [GvP80, VII.§1] imply that for $p > 5$ and for a Mumford curve X, if $\mathrm{Aut}(X)\backslash X \cong \mathbb{P}^1_K$, then the quotient map $X \to \mathbb{P}^1_K$ ramifies above at least 4 points. Applying the classical Hurwitz formula, we therefore see that $|\mathrm{Aut}(X)| \leq 12(g-1)$ (where g is the genus of X), which partly recovers the Herrlich's result [Her80b].

(4) Our list of p-adic triangle groups has a non-empty intersection with André's

list of p-adic arithmetic triangle groups, but does not include it, since André's p-adic arithmetic triangle groups are not of Mumford type in general. In other words, André's p-adic arithmetic triangle groups do not always come out with the uniformization by the spaces of form $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} - \mathcal{L}_\Gamma$, but by étale coverings of them. The following arithmetic triangle groups do not appear in our list:

$p = 5$: $(2, 5, 10)$, $(5, 5, 5)$, $(2, 15, 30)$, $(3, 10, 30)$, $(15, 15, 15)$.
$p = 3$: $(3, 6, 6)$, $(6, 12, 12)$, $(9, 18, 18)$.
$p = 2$: $(2, 4, 8)$, $(2, 8, 8)$, $(4, 4, 4)$, $(4, 8, 8)$, $(3, 4, 12)$, $(2, 8, 16)$, $(4, 16, 16)$, $(8, 8, 8)$, $(2, 12, 24)$, $(3, 8, 24)$, $(6, 24, 24)$, $(12, 12, 12)$.

1.3. *Outline of the proof.* The proof of the theorem will be carried out by studying the action on (a subtree of) the Bruhat-Tits tree (cf. [Mum72][GvP80][Ser80]) by discrete subgroups in $\mathrm{PGL}(2, \mathbb{C}_p)$. To a finitely generated discrete subgroup Γ, there exists associated tree $\mathcal{T}_\Gamma$ (resp. $\mathcal{T}^*_\Gamma$), which is the tree generated by the set of limit points (resp. limit points together with fixed points of elliptic elements) in $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$, being as the set of ends (see 2.1). The tree $\mathcal{T}_\Gamma$ is equal to the Mumford's tree (e.g. if Γ is a Schottky subgroup), but the other tree is in general "bigger". The advantage of the tree $\mathcal{T}^*_\Gamma$ is its link with the ramification (or, branch) points; more precisely, there exists a canonical bijection between branch points of the uniformization map $\varpi_\Gamma\colon (\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} - \mathcal{L}_\Gamma) \rightarrow \Gamma\backslash(\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} - \mathcal{L}_\Gamma)$ and ends of the quotient graph $T^*_\Gamma = \Gamma\backslash\mathcal{T}^*_\Gamma$ (Proposition 2.2). By this, we have the following principle:

Proposition. *A finitely generated discrete subgroup $\Gamma \subset \mathrm{PGL}(2, \mathbb{C}_p)$ is a p-adic triangle group of Mumford type if and only if the graph T^*_Γ is a tree having precisely three ends.* $\square$

That the graph T^*_Γ is a tree is equivalent to that the quotient $\Gamma\backslash(\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} - \mathcal{L}_\Gamma)$ is a curve of genus 0. The formation of the trees $\mathcal{T}^*_\Gamma$ admits the following obvious functoriality: For an inclusion $\Gamma_1 \subseteq \Gamma_2$ of finitely generated subgroups, we have an inclusion of trees $\mathcal{T}^*_{\Gamma_1} \subseteq \mathcal{T}^*_{\Gamma_2}$, and hence, a map $T^*_{\Gamma_1} \rightarrow T^*_{\Gamma_2}$.

We decorate the tree T^*_Γ by groups attached to vertices and edges, which are simply the stabilizers of them. This gives rise to the data, so called, the *tree of groups* $(T^*_\Gamma, \Gamma_\bullet)$. The main point of the proof is that, essentially, the data $(T^*_\Gamma, \Gamma_\bullet)$, considered as abstract tree of groups, recovers Γ. Needless to say, to recover Γ, one has to take nice embeddings of groups $\Gamma_\bullet$ in $\mathrm{PGL}(2, K)$. This has been discuss in [Kat00], where a complete criterion for an abstract tree of groups $(T, G_\bullet)$ to be realizable was given. By this, the task in the proof reduces basically to a purely combinatorial problem: Classify all possible $*$-admissible tree of groups with exactly three ends. This combinatorial business is easy in principle, but requires a lot of care. We will introduce a notion of push-out, or direct limit of trees of groups, which will be helpful to carry out the combinatorics. In proving the theorem, we will exhibit all the trees of groups in simple pictures, by which, besides, the abstract group structure of the corresponding Γ can be deduced.

1.4. *Notation and conventions.* Throughout this paper K denotes a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_K$ the integer ring, and π a prime element in $\mathcal{O}_K$. We write $[K\colon\mathbb{Q}_p] = ef$, where e is the ramification degree and $q = p^f$ is the the number

of elements in the residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$. We denote by $\nu\colon K^\times \to \mathbb{Z}$ the normalized (i.e., $\nu(\pi) = 1$) valuation.

For an abstract tree T we denote by $\mathrm{Vert}(T)$ (resp. $\mathrm{Edge}(T)$, $\mathrm{Ends}(T)$) the set of all vertices (resp. unoriented edges, ends). The notation $v \vdash \sigma$ for $v \in \mathrm{Vert}(T)$ and $\sigma \in \mathrm{Edge}(T)$ means that σ emanates from v. For a vertex $v \in \mathrm{Vert}(T)$ we denote by $\mathrm{Star}_v(T)$ the set of edges in $\mathrm{Edge}(T)$ emenating from v. For two vertices v_0 and v_1, we denote by $[v_0, v_1]$ the geodesic path connecting them. For $\varepsilon_0, \varepsilon_1 \in \mathrm{Ends}(T)$ and $v \in \mathrm{Vert}(T)$, the unique straight-line (resp. half-line) connecting ε_0 and ε_1 (resp. v and ε_0) is denoted by $]\varepsilon_0, \varepsilon_1[$ (resp. $[v, \varepsilon_0[$). The geometric realization $|T|$ is metrized so that the path $[v_0, v_1]$ ($v_0, v_1 \in \mathrm{Vert}(T)$) is of length equal to the number of edges in it. The metric function is denoted by $d_T(\cdot,\cdot)$, or simply by $d(\cdot,\cdot)$. If T is a subtree of $\mathfrak{T}_K$, the Bruhat-Tits tree attached to $\mathrm{PGL}(2, K)$, then we always regard the set $\mathrm{Ends}(T)$ as a subset of $\mathbb{P}^1(K)$. In dealing with a tree we often switch to regard it as a topological space by means of the geometric realization.

2. Preliminaries

Let us first review the basic facts on trees and groups which were dealt with in [Kat00]:

2.1. *Trees and groups* (cf. [Kat00, §2]). Let Γ be a finitely generated discrete subgroup in $\mathrm{PGL}(2, K)$, and suppose that K has been chosen to be large enough such that the fixed points (in $\mathbb{P}^1_K$) of any elliptic element in Γ are K-valued. Such a Γ associates a subtree $\mathfrak{T}^*_\Gamma$ in the Bruhat-Tits tree $\mathfrak{T}_K$ characterized by (i) the set of ends of $\mathfrak{T}^*_\Gamma$ are in the canonical bijection with the closure of the set of fixed points of each element ($\neq 1$) in Γ, and (ii) $\mathfrak{T}^*_\Gamma$ is minimal among subtrees having this property. Clearly, $\mathfrak{T}^*_\Gamma$ is acted on by Γ. Attaching the stabilizers to each vertex and edge, we get the tree of groups $(\mathfrak{T}^*_\Gamma, \Gamma_\bullet)$. Suppose that the quotient $T^*_\Gamma = \Gamma\backslash\mathfrak{T}^*_\Gamma$ is a tree, by which one can consider a section $\iota_\Gamma\colon T^*_\Gamma \hookrightarrow \mathfrak{T}^*_\Gamma$. The section ι_Γ gives rise to a tree of groups $(T^*_\Gamma, \Gamma_\bullet)$ in the obvious way.

Convention. Throughout this paper, when a finitely generated discrete subgroup Γ is discussed, the field K is assumed to be chosen large enough so that the tree $\mathfrak{T}^*_K$ can be defined.

2.2. Proposition. *There exist canonical bijections, compatible with the quotient maps,*

$$\begin{array}{ccc}
\left\{\begin{array}{l}\textit{Ramification points in}\\ \Omega_\Gamma \textit{ of the map } \varpi_\Gamma\end{array}\right\} & \longleftrightarrow & \mathrm{Ends}(\mathfrak{T}^*_\Gamma) - \mathrm{Ends}(\mathfrak{T}_\Gamma) \\
\Big\downarrow & & \Big\downarrow \\
\{\textit{Branch points in } \Gamma\backslash\Omega_\Gamma\} & \longleftrightarrow & \mathrm{Ends}(T^*_\Gamma).
\end{array}$$

Moreover, the decomposition group of a ramification point coincides with the stabilizer of the corresponding end. $\square$

2.3. $*$*-admissibility* ([Kat00, §3]). Conversely, let $(T, G_\bullet)$ be an abstract tree of groups, and suppose that we are given embeddings $G_v \hookrightarrow \mathrm{PGL}(2, K)$ for any $v \in \mathrm{Vert}(T)$ compatible with each $G_\sigma \hookrightarrow G_v$ for any $v \vdash \sigma$. Such embeddings gives rise to subtrees $\mathfrak{T}^*_{G_v}$ (for $v \in \mathrm{Vert}(T)$) as in 2.1. Let $\widetilde{\mathfrak{T}}_{G_\bullet}$ be the minimal subtree in

$\mathfrak{T}_K$ containing all $\mathfrak{T}^*_{G_v}$. The set of ends in $\widetilde{\mathfrak{T}}_{G_\bullet}$ is the union of the set of ends in $\mathfrak{T}^*_{G_v}$ for $v \in \mathrm{Vert}(T)$. This tree is labelled by groups (not necessarily finite) $\widetilde{G}_\bullet$ as follows: For a vertex $v \in \mathrm{Vert}(\widetilde{\mathfrak{T}}_{G_\bullet})$ the group $\widetilde{G}_v$ is the subgroup in $\mathrm{PGL}(2,K)$ generated by $(G_u)_v$ (the stabilizer at v by the action of G_u on $\mathfrak{T}_K$) for all $u \in \mathrm{Vert}(T)$; the definition of the group $\widetilde{G}_\sigma$ for $\sigma \in \mathrm{Edge}(\widetilde{\mathfrak{T}}_{G_\bullet})$ is similar, which is just the intersection of $\widetilde{G}_v$'s at the two extremities.

2.4. Definition. An *admissible embedding* of an abstract tree of groups $(T, G_\bullet)$ is an embedding $\iota\colon T \hookrightarrow \mathfrak{T}_K$ of trees together with embeddings $G_v \hookrightarrow \mathrm{PGL}(2,K)$ for any $v \in \mathrm{Vert}(T)$ compatible with each $G_\sigma \hookrightarrow G_v$ for any $v \vdash \sigma$ such that the following conditions are satisfied:

1. $\iota(T) \subset \widetilde{\mathfrak{T}}_{G_\bullet}$.
2. For any $v \in \mathrm{Vert}(T)$ and $\gamma \in G_v$ ($\gamma \neq 1$), there exists $\delta \in \Gamma$ such that $M(\delta\gamma\delta^{-1}) \bigcap \iota(T)$ contains an edge, where Γ is the subgroup in $\mathrm{PGL}(2,K)$ generated by all G_v for $v \in \mathrm{Vert}(T)$.
3. $\widetilde{G}_{\iota(v)} = G_v$ for any $v \in \mathrm{Vert}(T)$.
4. $\widetilde{G}_{\iota(\sigma)} = G_\sigma$ for any $\sigma \in \mathrm{Edge}(T)$.
5. For any $v \in \mathrm{Vert}(T)$, we have $\mathrm{Star}_v(T) \cong G_v \backslash (G_v \cdot \mathrm{Star}_{\iota(v)}(\widetilde{\mathfrak{T}}_{G_\bullet}))$ by the composite of ι followed by the projection.

An abstract tree of groups $(T, G_\bullet)$ is said to be $*$-*admissible* if it has an admissible embedding and the associated amalgam $\lim_{\rightarrow}(T, G_\bullet)$ is finitely generated.

The tree of groups $(T^*_\Gamma, \Gamma_\bullet)$ associated to a finitely generated discrete subgroup Γ is $*$-admissible by the section ι_Γ (cf. [Kat00, 3.5]). The following theorem, proved in [Kat00, §3], states the converse:

2.5. Theorem. *Let $(T, G_\bullet)$ be a $*$-admissible tree of groups and $\iota\colon T \hookrightarrow \widetilde{\mathfrak{T}}_{G_\bullet}$ with $\{G_v \hookrightarrow \mathrm{PGL}(2,K)\}_{v\in\mathrm{Vert}(T)}$ an admissible embedding. Let Γ be the subgroup in $\mathrm{PGL}(2,K)$ generated by all G_v for $v \in \mathrm{Vert}(T)$ and set*

$$\mathfrak{T}^* = \bigcup_{\gamma\in\Gamma} \gamma \cdot \iota(T)$$

in $\mathfrak{T}_K$. Then:

(1) The group Γ is a finitely generated discrete subgroup in $\mathrm{PGL}(2,K)$ isomorphic to $\lim_{\rightarrow}(T, G_\bullet)$.

(2) The subset $\mathfrak{T}^*$ in $\mathfrak{T}_K$ is a tree and $\Gamma\backslash\mathfrak{T}^* \cong T$.

(3) The embedding ι gives a section $T \hookrightarrow \mathfrak{T}^*$ by which the induced tree of groups $(T, \Gamma_\bullet)$ equals to $(T, G_\bullet)$.

Moreover, if $\mathfrak{T}^*_\Gamma$ is the tree associated to Γ as in 2.1, then $\mathfrak{T}^*_\Gamma \subseteq \mathfrak{T}^*$, and the induced inclusion $T^*_\Gamma \hookrightarrow T$ enjoys the following:

(4) The induced inclusion $\mathrm{Ends}(T^*_\Gamma) \hookrightarrow \mathrm{Ends}(T)$ is a bijection.

(5) The section ι restricts to a section $T^*_\Gamma \hookrightarrow \mathfrak{T}^*_\Gamma$ by which the induced tree of groups $(T^*_\Gamma, \Gamma_\bullet)$ is the restriction of $(T, \Gamma_\bullet) = (T, G_\bullet)$.

(6) The tree of groups $(T^*_\Gamma, \Gamma_\bullet)$ is a contraction of $(T, \Gamma_\bullet) = (T, G_\bullet)$. $\square$

2.6. Lemma. *Let Γ be a finitely generated discrete subgroup such that $T^*_\Gamma = \Gamma\backslash\mathcal{T}^*_\Gamma$ is contraction minimal, i.e., there is no proper subtree in T^*_Γ having the same set of ends.*

Proof. Otherwise, there exists a vertex $v \in \mathrm{Vert}(T^*_\Gamma)$ such that $T - \{v\}$ is connected. Hence such a vertex occurs also in $\mathcal{T}^*_\Gamma$; but this contradicts that $\mathcal{T}^*_\Gamma$ is the smallest one having the prescribed set of ends. $\square$

2.7. Definition. A *tripod* is a tree which is the union of three half-lines ℓ_i $(i = 1, 2, 3)$ starting at a common vertex v, called the *center*, such that $\ell_i \bigcap \ell_j = \{v\}$ for any $i \neq j$.

2.8. Corollary. *If Γ is a p-adic Schwarzian triangle groups of Mumford type, then T^*_Γ is a tripod.*

Proof. Clear from Proposition 1.3 and Lemma 2.6. $\square$

2.9. Finally, we recall the following: Let γ $(\neq 1)$ be an elliptic element of finite order in $\mathrm{PGL}(2, K)$, and suppose K is taken to be large enough for the fixed points of γ to be K-valued. The apartment connecting two fixed points of γ is called the *mirror* of γ, and denoted by $M(\gamma)$. Needless to say, it is contained in the fixed locus by γ in $\mathcal{T}_K$.

2.10. Lemma ([Kat00, 2.10]). *Let n be the order of γ, and set $G = \langle\gamma\rangle$.*

(1) Let $v_0 \in M(\gamma)$. If $(n, p) = 1$, then G acts freely on the $q-1$ vertices adjacent to v_0 not lying on $M(\gamma)$, where q is the number of elements in the residue field k.

(2) Suppose $n = p^r$ for $r \geq 1$, and set $s = \nu(\zeta_{p^r} - 1)$, where ζ_{p^r} is a primitive p^r-th root of unity, and ν is the normalized (i.e. $\nu(\pi) = 1$) valuation. Then a vertex $v \in \mathcal{T}_K$ is fixed by G if and only if $0 \leq d(v, M(\gamma)) \leq s$. $\square$

3. Direct limit of trees

3.1. Given a diagram $\mathcal{T}_1 \leftarrow \mathcal{T}_0 \rightarrow \mathcal{T}_2$ of morphisms of trees, one can define the push-out

$$\begin{array}{ccc} \mathcal{T}_0 & \longrightarrow & \mathcal{T}_1 \\ \big\downarrow & & \big\downarrow \\ \mathcal{T}_2 & \longrightarrow & \mathcal{T}_1 \sharp_{\mathcal{T}_0} \mathcal{T}_2 \end{array}$$

in the category of graphs; it is, regarded as a diagram of topological spaces, simply the push-out in the category of topological spaces. In slightly more formal terms, the graph $\mathcal{T}_1 \sharp_{\mathcal{T}_0} \mathcal{T}_2$ has the set of vertices $\mathrm{Vert}(\mathcal{T}_1) \coprod_{\mathrm{Vert}(\mathcal{T}_0)} \mathrm{Vert}(\mathcal{T}_2)$ (push-out of sets) and the similarly defined set of oriented edges together with the naturally defined notion of origin and terminus of edges. It is clear that the push-out $\mathcal{T}_1 \sharp_{\mathcal{T}_0} \mathcal{T}_2$ of trees is again a tree, provided that $\mathcal{T}_0$ is not empty.

The similar construction can be applied for push-out of trees of groups, where a morphism $\phi\colon (\mathcal{T}_0, G_{0,\bullet}) \rightarrow (\mathcal{T}_1, G_{1,\bullet})$ of trees of groups is defined to be a morphism of trees $\phi\colon \mathcal{T}_0 \rightarrow \mathcal{T}_1$ together with the collection of monomorphisms of groups $G_* \rightarrow G_{\phi(*)}$. For a diagram $(\mathcal{T}_1, G_{1,\bullet}) \leftarrow (\mathcal{T}_0, G_{0,\bullet}) \rightarrow (\mathcal{T}_2, G_{2,\bullet})$ of trees of groups, the

push-out $(\mathcal{T}_1\sharp_{\mathcal{T}_0}\mathcal{T}_2, G_{1_02,\bullet})$ is endowed with the amalgam groups; to see that it is actually a tree of groups, one has to show that for $v \vdash \sigma$ in $\mathcal{T}_1\sharp_{\mathcal{T}_0}\mathcal{T}_2$ the induced morphism $G_{1_02,\sigma} \to G_{1_02,v}$ is injective. This follows from the structure theorem of amalgam groups [Ser80, I.1.2].

3.2. Example. Let $\mathcal{T}_0$ be the straight-line

$$\mathcal{T}_0 = \cdots — v_{-2} — v_{-1} — v_0 — v_1 — v_2 — \cdots,$$

and $\mathcal{T}_1$ and $\mathcal{T}_2$ half-lines

$$\begin{aligned} \mathcal{T}_1 &= u_0 — u_1 — u_2 — \cdots, \\ \mathcal{T}_2 &= \cdots — w_{-2} — w_{-1} — w_0. \end{aligned}$$

Let m be a positive integer. The morphisms $f\colon \mathcal{T}_0 \to \mathcal{T}_1$ and $g\colon \mathcal{T}_0 \to \mathcal{T}_2$ are defined by $f(v_n) = u_{|n|}$ and $g(v_n) = w_{-|n-m|}$, respectively. Then the push-out $\mathcal{T}_1\sharp_{\mathcal{T}_0}\mathcal{T}_2$ is a segment of length m, isomorphic to $[u_0, u_m]$ (and $[w_{-m}, w_0]$).

3.3. Let Γ a finitely generated discrete subgroup (e.g. a finite subgroup) in $\mathrm{PGL}(2,K)$, and $G \subseteq \Gamma$ a finite subgroup. By the construction of $*$-trees (2.1, cf. [Kat00, §2]), we have an inclusion of subtrees $\mathcal{T}^*_G \subseteq \mathcal{T}^*_\Gamma$, which yields the commutative diagram

$$\begin{tikzcd} \mathcal{T}^*_G \arrow[r, hook] \arrow[d, "\varrho_G"'] & \mathcal{T}^*_\Gamma \arrow[d, "\varrho_\Gamma"] \\ T^*_G \arrow[r, "\varrho^G_\Gamma"'] & T^*_\Gamma \end{tikzcd}$$

of graphs, where ϱ_G and ϱ_Γ are quotients by G and Γ, respectively. The morphism ϱ^G_Γ is not in general injective. Note that (as one can see in Appendix below) the quotient graph T^*_G for a finits subgroup is always a tree. Suppose that T^*_Γ is a tree. The four trees in the above diagram are then endowed with finite groups as in 2.1, and become trees of groups. The above diagram has an obvious extension to a diagram of trees of groups, where the morphism between attached groups are defined abstractly.

Now suppose that we are given two finite subgroup G_1 and G_2 in $\mathrm{PGL}(2,K)$ with $G_0 = G_1 \bigcap G_2$ such that $G_0 \neq \{1\}$ Then one can consider the push-out diagram

$$\begin{tikzcd} T^*_{G_0} \arrow[r, "\varrho^{G_0}_{G_1}"] \arrow[d, "\varrho^{G_0}_{G_2}"'] & T^*_{G_1} \arrow[d] \\ T^*_{G_2} \arrow[r] & T^*_{G_1}\sharp_{T^*_{G_0}} T^*_{G_2} \end{tikzcd}$$

of abstract trees of groups.

3.4. Lemma. *The following conditions are equivalent:*

(1) *For any $v \in \mathrm{Vert}(T^*_{G_1}\sharp_{T^*_{G_0}} T^*_{G_2})$, the group attached to v is finite.*
(2) *For any $v \in \mathrm{Vert}(\mathcal{T}^*_{G_0})$, either $G_{1,v} \subseteq G_{2,v}$ or $G_{2,v} \subseteq G_{1,v}$ holds.*

Proof. Clear by the construction of push-out of trees of groups and the structure theorem of amalgam groups [Ser80, I.1.2]. □

3.5. Let Γ be a finitely generated discrete subgroup in $\mathrm{PGL}(2,K)$ such that T^*_Γ is a tree, and $\iota_\Gamma\colon T^*_\Gamma \hookrightarrow \mathfrak{T}^*_\Gamma$ a section. The construction as in 3.3 can be carried out for Γ_{v_1} and Γ_{v_2} with the intersection Γ_σ coming from each edge σ with extremities v_1 and v_2 in T^*_Γ as far as $\Gamma_\sigma \neq \{1\}$. Hence we can define the *direct limit* along T^*_Γ (similarly as in [Ser80, I.1.2, below Theorem 2]), denoted by

$$\varinjlim T^*_{\Gamma_\bullet},$$

which is a disjoint union of trees of groups; it is a single tree of group, if there is no edge in T^*_Γ with trivial group. The morphisms $\varrho_\Gamma^{\Gamma_v}$ ($v \in \mathrm{Vert}(T^*_\Gamma)$) and $\varrho_\Gamma^{\Gamma_\sigma}$ ($\sigma \in \mathrm{Edge}(T^*_\Gamma)$) induce a morphism

$$\varinjlim T^*_{\Gamma_\bullet} \longrightarrow T^*_\Gamma. \tag{3.5.1}$$

It is clear that the image S of this morphism is the union of the images of $\mathfrak{T}^*_{\Gamma_v}$ for all $v \in \mathrm{Vert}(T^*_\Gamma)$ under the quotient map ϱ_Γ. Let $S = \coprod_{i\in I} S_i$ be the decomposition into connected components, and s_{ij} ($i,j \in I$, $i \neq j$) the geodesic path connecting S_i and S_j.

3.6. Lemma. *If s_{ij} does not meet any S_k for $k \notin \{i,j\}$, then it contains an edge σ with $\Gamma_\sigma = \{1\}$.*

Proof. By Lemma 2.10, there exist two increasing sequences of p-subgroups consisting of subgroups in the stabilizers of vertices in s_{ij}, which are increasing ordered approaching to each S_i and S_j. If there is no σ with $\Gamma_\sigma = \{1\}$, then there exists a vertex v in s_{ij} fixed by two non-trivial p-groups having distinct mirrors. If $v \neq v_1$ and $v \neq v_2$, then, Γ_v is not contained in Γ_w's for any vertex w in S, and hence, v is in the image of $\mathfrak{T}^*_{\Gamma_v}$, thereby the contradiction. If $v \in S_i$, then $\mathfrak{T}^*_{\Gamma_v}$ contains a mirror which is mapped to S_j, and hence s_{ij} is in the image of $\mathfrak{T}^*_{\Gamma_v}$. □

3.7. Lemma. *Let $(T, G_\bullet)$ be a $*$-admissible tree of groups, and $\sigma \in \mathrm{Edge}(T)$ an edge such that $G_\sigma = \{1\}$. Decompose $T = T_1 \bigcup [v_1, v_2] \bigcup T_2$, where v_1 and v_2 are the extremities of σ, such that $T_i \bigcap [v_1, v_2] = \{v_i\}$ for $i = 1, 2$. Then each $(T_i, G_\bullet)$ is $*$-admissible, and $\Gamma \cong \Gamma_1 * \Gamma_2$, where $\Gamma_i = \lim_\rightarrow (T_i, G_\bullet)$.*

Proof. Clearly, we have $\Gamma \cong \Gamma_1 * \Gamma_2$. Consider an admissible embedding of $(T, G_\bullet)$, which is restricted to each T_i. To show that $(T_1, G_\bullet)$ is $*$-admissible, only the condition (2.4.2) calls for a verification. Let $v \in \mathrm{Vert}(T_1)$ and $\gamma \in G_v - \{1\}$. Take $\delta \in \Gamma$ such that $M(\delta\gamma\delta^{-1}) \bigcap T \neq \emptyset$. If $M(\delta\gamma\delta^{-1}) \bigcap T_2 \neq \emptyset$, then there exists $w \in \mathrm{Vert}(T_2)$ such that $\chi = \delta\gamma\delta^{-1}$ belongs to G_w yielding a non-trivial relation between elements in Γ_1 and Γ_2. Hence $M(\delta\gamma\delta^{-1}) \bigcap T_1 \neq \emptyset$, thereby the lemma. □

3.8. Definition. A tree of groups $(T, G_\bullet)$ is said to be *irreducible* if T does not contain an edge to which the trivial group is attached.

Due to Lemma 3.6 and the minimality of T^*_Γ (Lemma 2.6), if T^*_Γ is irreducible, then the map (3.5.1) of trees is surjective, i.e. $S = T^*_\Gamma$.

3.9. Proposition. *If $(T^*_\Gamma, \Gamma_\bullet)$ is irreducible, then the morphism (3.5.1) is an isomorphism of trees of groups. In general, it is injective, and maps every connected component of $\lim_{\longrightarrow} T^*_{G_\bullet}$ isomorphically onto a subtree of groups in T^*_Γ.*

Before the proof, we need:

3.10. Definition. Let G_1 and G_2 be subgroups in $\mathrm{PGL}(2, K)$. Then we say that G_1 and G_2 are *in regular position* if, for any K-split torus T in $\mathrm{PGL}(2, K)$, $G_1 \bigcap T \neq \{1\}$ and $G_2 \bigcap T \neq \{1\}$ imply $G_1 \bigcap G_2 \bigcap T \neq \{1\}$.

3.11. Lemma. *Let $v_1, v_2 \in \mathrm{Vert}(T^*_\Gamma)$. Then Γ_{v_1} and Γ_{v_2} are in regular position.*

Proof. Let T be a split torus, and $\gamma_1 \in \Gamma_{v_1} \bigcap T - \{1\}$ and $\gamma_2 \in \Gamma_{v_2} \bigcap T - \{1\}$. Let $s = [v_1, v_2]$. Then $\Gamma_{v_1} \bigcap \Gamma_{v_2}$ is the set of elements in Γ which fix s pointwise. If the mirror $M = M(\gamma_1) = M(\gamma_2)$ contains s, then $\gamma_1 \in \Gamma_{v_1} \bigcap \Gamma_{v_2}$. If not, we have two cases: First, if M does not meet the interior of s, then, exchanging indices if necessary, we may assume that v_1 is nearer to M than v_2. Then s is fixed by γ_2, and hence, $\gamma_2 \in \Gamma_\sigma = \Gamma_{v_1} \bigcap \Gamma_{v_2}$. If M meets s at a interior vertex v, then, by Lemma 2.10, γ_1 and γ_2 are p-elements, and hence, exchanging indices if necessary, we may assume $\langle\gamma_1\rangle \subseteq \langle\gamma_2\rangle$. In this case, s is fixed by γ_2. $\square$

For a finitely generated discrete subgroup Γ, $\mathcal{F}_\Gamma$ denotes the set of points in $\mathbb{P}^1_K$ fixed by an element in $\Gamma - \{1\}$. Due to our convension about the field K, it consists of K-valued points.

3.12. Lemma. *Let $\Gamma_1, \Gamma_2 \subset \mathrm{PGL}(2, K)$ be finitely generated discrete subgroups which are in regular position. Then we have $\mathcal{F}_{\Gamma_1 \bigcap \Gamma_2} = \mathcal{F}_{\Gamma_1} \bigcap \mathcal{F}_{\Gamma_2}$. In particular, if Γ_1 and Γ_2 are finite, then we have $\mathrm{Ends}(\mathcal{T}^*_{\Gamma_1 \bigcap \Gamma_2}) = \mathrm{Ends}(\mathcal{T}^*_{\Gamma_1} \bigcap \mathcal{T}^*_{\Gamma_2})$.*

Proof. $\mathcal{F}_{\Gamma_1 \bigcap \Gamma_2} \subseteq \mathcal{F}_{\Gamma_1} \bigcap \mathcal{F}_{\Gamma_2}$ is clear. Let $z \in \mathcal{F}_{\Gamma_1} \bigcap \mathcal{F}_{\Gamma_2}$. There exist $\gamma_1 \in \Gamma_1 - \{1\}$ and $\gamma_2 \in \Gamma_2 - \{1\}$ such that $\gamma_1(z) = \gamma_2(z) = z$. Since no two element in a discrete subgroup share exactly one fixed point (well-known, cf. [Kat00, 2.5]), γ_1 and γ_2 belong to a same split torus. By the assumption, there exists $\gamma_3 \in \Gamma_1 \bigcap \Gamma_2 - \{1\}$ having the same fixed points as γ_1 and γ_2, thereby the lemma. $\square$

Proof of Proposition 3.9. We embedd T^*_Γ into $\mathcal{T}^*_\Gamma$ (together with attached groups) by the section ι_Γ fixed at the beginning of our construction. Let S be the intersection of T^*_Γ and the union of all $\mathcal{T}^*_{\Gamma_v}$ for $v \in \mathrm{Vert}(T^*_\Gamma)$, which coincides with the image of the union of all $\mathcal{T}^*_{\Gamma_v}$ under the quotient map. We attach groups to S in the obvious way. For any $v \in \mathrm{Vert}(T^*_\Gamma)$, consider the natual morphism $T^*_\Gamma \bigcap \mathcal{T}^*_{\Gamma_v} \to \lim_{\longrightarrow} T^*_{\Gamma_\bullet}$. We can glue thus obtained morphisms to a morphism (together with morphisms of groups) defined on S; indeed, for $v_1, v_2 \in \mathrm{Vert}(T^*_\Gamma)$, Lemma 3.11, Lemma 3.12, and Lemma 2.6 imply that $T^*_\Gamma \bigcap \mathcal{T}^*_{\Gamma_{v_1}} \bigcap \mathcal{T}^*_{\Gamma_{v_2}} = T^*_\Gamma \bigcap \mathcal{T}^*_{\Gamma_{v_1} \bigcap \Gamma_{v_2}}$. It is easily verified that this morphism on S gives the inverse of (3.5.1). $\square$

4. Construction of triangle groups

In this section, we will construct p-adic Schwarzian triangle groups of Mumford type in $p = 2, 3, 5$. It will be proved in the next section that these are actually the only possible triangle groups.

4.1. $p = 5$. First we discuss in $p = 5$, i.e., K is a finite extention of $\mathbb{Q}_5$. We begin with a finite subgroup $G_1 \subset \mathrm{PGL}(2, K)$ isomorphic to A_5. Inside G_1 we consider a subgroup $G_0 \subset G_1$ isomorphic to D_5. The morphism $\varrho_1^0 : T^*_{G_0} \to T^*_{G_1}$ is described as follows (see A.9 and A.12 in Appendix; the pictures are drawn obeying the convention in A.7):

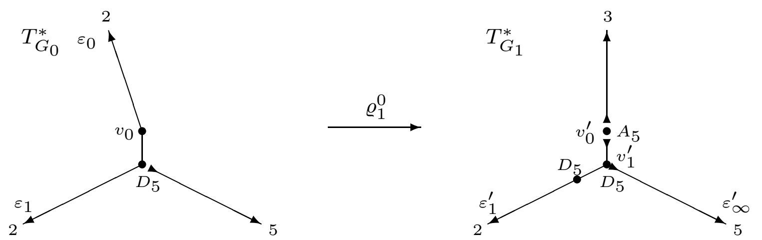


Figure 1

Here, ϱ_1^0 maps the locus below v_0 into the locus below v_0'. The straight-line $]\varepsilon_0, \varepsilon_1[$ is mapped to the half-line $[v_0', \varepsilon_1'[$ by a map like $x \mapsto |x|$ with the folding at v_0.

Next we consider a finite subgroup G_2 isomorphic to D_{10m} ($m \geq 1$) such that $G_0 \subseteq G_2$; G_2 is generated by G_0 and an element of order $10m$ which commutes with elements of order 5 in G_0. The morphism $\varrho_2^0 : T^*_{G_0} \to T^*_{G_2}$ is described as follows:

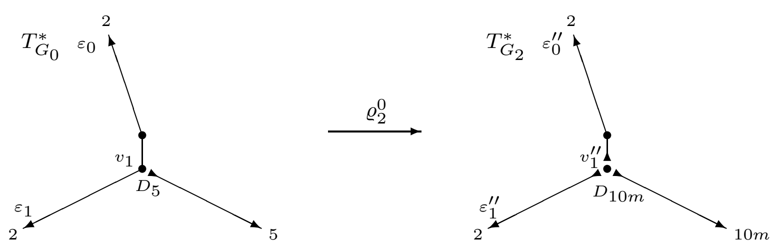


Figure 2

Here the straight line $]\varepsilon_0, \varepsilon_1[$ is mapped to the half-line $[v_1'', \varepsilon_0''[$. By means of these data, it is not difficult to compute the push-out $T = T^*_{G_1} \sharp_{T^*_{G_0}} T^*_{G_2}$; the only point to pay attention is that the straight line $]\varepsilon_0, \varepsilon_1[$ is mapped in T onto a segment isomorphic to $[v_0, v_1]$ by the same reasoning as in Example 3.2. The resulting tree of groups T looks like as in Figure 3.

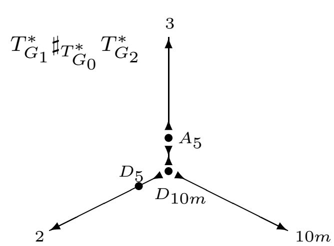


Figure 3

4.2. We are going to show that the tree of groups $T = T^*_{G_1} \sharp_{T^*_{G_0}} T^*_{G_2}$ is $*$-admissible. Let $\iota_{G_1}\colon T^*_{G_1} \to \mathfrak{T}^*_{G_1}$ be a section. Then we can find a section $\iota_{G_2}\colon T^*_{G_2} \to \mathfrak{T}^*_{G_2}$ such that $\iota_{G_1}(T^*_{G_1}) \bigcap \iota_{G_2}(T^*_{G_2})$ is the half-line $[\iota_{G_1}(v'_0), \iota_{G_1}(\varepsilon'_\infty)[$. Indeed, $\iota_{G_2}(T^*_{G_2})$ is the union of the following three half-lines: (i) $[\iota_{G_1}(v'_1), \iota_{G_1}(\varepsilon'_\infty)[$, (ii) a half-line in a mirror of an element of order 2 in G_1 containing $[\iota_{G_1}(v'_0), \iota_{G_1}(v'_1),]$ and starting at $\iota_{G_1}(v'_1)$, and (iii) a half-line starting at $\iota_{G_1}(v'_1)$ contained in a mirror of an element of order 2 in G_2 not in G_0 and $Z_{10m} \subset G_2$. Let us define $T^*_{G_1} \to \iota_{G_1}(T^*_{G_1})$ (resp. $T^*_{G_2} \to \iota_{G_2}(T^*_{G_2})$) which coincides with ι_{G_1} (resp. ι_{G_2}) except for that the half-line $[v'_0, \varepsilon'_1]$ (resp. $[v''_1, \varepsilon''_0]$) is mapped to $[\iota_{G_1}(v'_0), \iota_{G_1}(v'_1)]$ in an obvious way. This induces an embedding $T \to \mathfrak{T}_K$ of the tree T. Then, together with $G_1, G_2 \hookrightarrow \mathrm{PGL}(2, K)$, this gives an admissible embedding; indeed, the local structure of T at any vertex is isomorphic to that around a vertex either in $T^*_{G_1}$ or in $T^*_{G_2}$, because our situation satisfies the condition (2) in Lemma 3.4. Hence (2.4.3), (2.4.4), and (2.4.5) are valid. The validity of (2.4.2) is evident, since T contains the images of all the mirrors in $T^*_{G_1}$ and $T^*_{G_2}$.

As a result, we get a triangle group of index $(2, 3, 10m)$ ($m \geq 1$), isomorphic to the amalgam product $A_5 *_{D_5} D_{10m}$.

4.3. We can also find the following two $*$-admissible trees of groups:

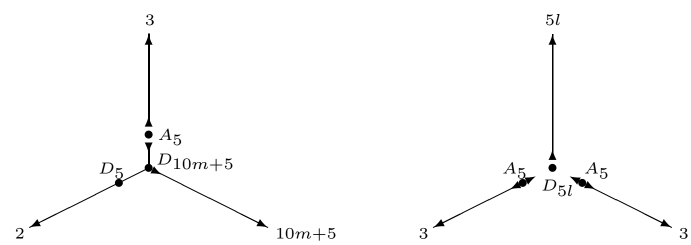


Figure 4

We sketch the construction of these trees, and details are left to the reader: The construction of the first one in Figure 4 is similar to that of Figure 3 as above; it is even simpler, because, in this case, the morphism $\varrho^0_2\colon T^*_{D_5} \to T^*_{D_{10m+5}}$ is injective. We call in general the procedure like this a *replacement*; it is, so to speak, the replacement of the mirror of order 5 by that of order $10m + 5$. The second one with $l = 1$ is by $T^*_{G_1} \sharp_{T^*_{G_0}} T^*_{G'_1}$, where G'_1 is another embedded A_5 which is the twist of G_1 by the non-trivial element in $N(D_5)/(N(D_5) \bigcap N(A_5)) \cong Z_2$. Here $N(G)$ for a subgroup $G \subseteq \mathrm{PGL}(2, K)$ stands for the normalizer of G in $\mathrm{PGL}(2, K)$; note that $N(D_5) = D_{10}$ and $N(A_5) = A_5$.

The second one with l odd is constructed by the method similar to that of the first one (replacement of the mirror of order 5 by that of order $10m + 5$), started by the one with $l = 1$. The construction of the second one with l even is outlined as follows: We start at the tree of groups T as in Fugure 3. Let δ be the involution in D_{10m} not contained in G_0 nor in $Z_{10m} \subset D_{10m}$. Let $G'_1 = \delta G_1 \delta$ and $G'_0 = \delta G_0 \delta$. (Note that the "D_5" denoted in Figure 3 is G'_0.) Then we consider the push-out $T \sharp_{T^*_{G'_0}} T^*_{G'_1}$, which can be described by a similar method as in 4.1, and this gives the desired tree of groups.

These are shown to be $*$-admissible by means of appropriate embeddings, constructed by an idea similar to that in 4.2.

In the following (until the end of this section), we will perform only sketchy constructions, but will present necessary data by which the reader can verify the details at each step; all the trees presented below are proved to be $*$-admissible by an appropriate embeddings which can be easily found, similarly as above.

4.4. $p = 3$. The arguement similar to that in $p = 5$, where (A_5, D_5) is replaced by (A_5, D_3) and (S_4, D_3), works in $p = 3$, which yields the following six $*$-admissible trees of groups $(m, l \geq 1)$

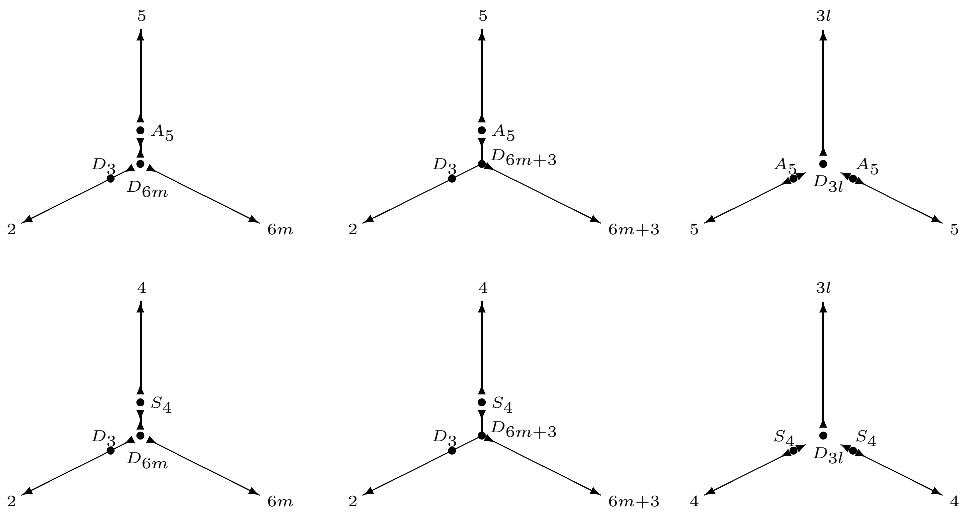


Figure 5

Also, it is easy to find that the method to obtain the two trees in the last column in Figure 5 is mixed up to get the one in Figure 6.

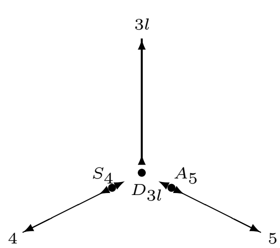


Figure 6

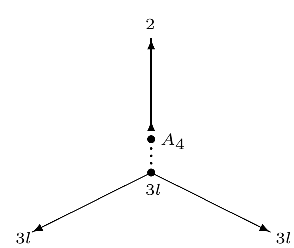


Figure 7

The tree in Figure 7 can be found by considering $T^*_{A_4} \sharp_{T^*_{Z_3}} T^*_{Z_{3l}}$, which is nothing but the replacement of the mirror of order 3 by that of order $3l$.

4.5. $p = 2$. There are plenty of triangle groups in $p = 2$. First, the push-out by

$(G_1, G_0, G_2) = (S_4, A_4, A_5)$ yields the first tree in Figure 8 with $l = 1$.

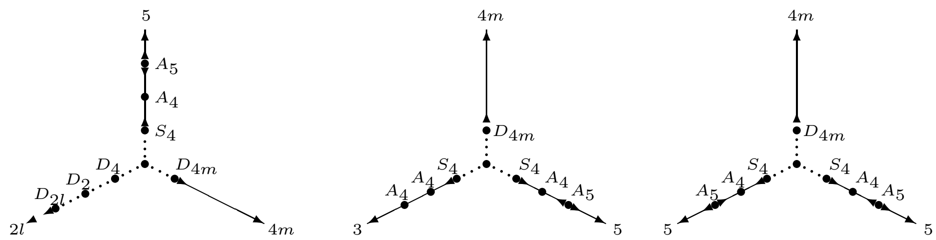


Figure 8

The third tree with $m = 1$ is obtained from the first one T (with $m = 1$) by the push-out $T\sharp_{T^*_{D_4}} T'$, where T' is the twist $T' = \delta T$ by a non-trivial element in $N(D_4)/(N(D_4) \bigcap N(S_4)) \cong Z_2$ (note that $N(D_4) = D_8$). The second one with $m = 1$ is $T\sharp_{T^*_{D_4}} \delta T^*_{G_1}$. Replacing the mirror of order 4 by that of order $4m$, we get the ones in $m \geq 1$. The center of the trees in Figure 8 is fixed by D_4 if m is odd, or by D_8 otherwise. Note that the, in the last two trees in Figure 8, the mirror of order 2 of each S_4 is absorbed in the mirror of order $4m$, which is easily seen by means of push-out.

In the first tree in Figure 8, the mirror of order 2 has been already replaced by that of order $2l$; but we claim that this replacement can be done if and only if l is odd. To see this, let G_1 be an embedded S_4, and $G_0 \subset G_1$ a subgroup isomorphic to D_2. The morphism $\varrho_1^0\colon T^*_{G_0} \to T^*_{G_1}$ is described as follows:

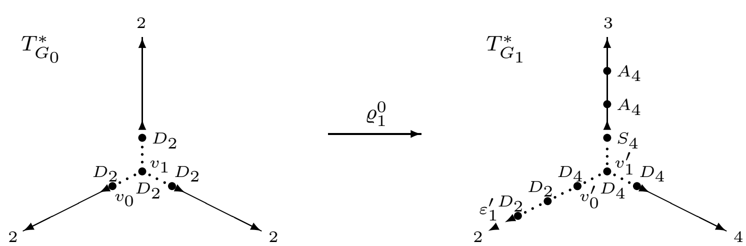


Figure 9

The map ϱ_1^0 maps each half-line starting at v_1 in $T^*_{G_0}$ to the half-line $[v_1', \varepsilon_1'[$. Let G_2 be isomorphic to D_{2l} with $G_1 \bigcap G_2 = G_0$ taken as in 4.1. Then the map ϱ_2^0 maps $T^*_{G_0}$ injectively onto $T^*_{G_2}$. The center of $T^*_{G_2}$ is fixed by D_2 if l is odd, or by D_4 otherwise. The center v_1 in $T^*_{G_0}$ is mapped to v_1' fixed by D_4; if l is even, then the conditon (2) in Lemma 3.4 is not satisfied at v_1, since these two D_4's at v_1' and the center in $T^*_{G_2}$ are not comparable. Hence l must be odd, and in this case, one can verify that the resulting tree of groups is $*$-admissible. Also, the last argument shows that the first tree in Figure 10 is $*$-admissible, which is simply obtained by

two replacements of mirrors.

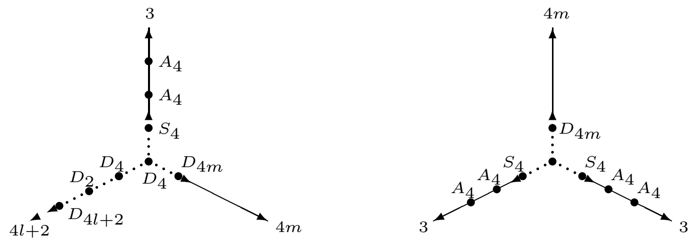


Figure 10

The tree in the right-hand side in Figure 10 with $m = 1$ is obtained by the push-out $T^*_{G_1} \sharp_{T^*_{G_0}} T^*_{G'_1}$, where $G_1 \cong S_4$, $G_0 \cong D_4$, and G'_1 is the twist of G_1 by the non-trivial element in $N(G_0)/(N(G_0) \bigcap N(G_1)) \cong Z_2$ (note that $N(D_4) = D_8$). By the suitable replacement of mirrors, we get the one with $m \geq 1$. The center is fixed by D_4 if m is odd, or D_8 otherwise.

Next consider $G_1 \cong A_5$ with a subgroup $G_0 \cong D_2$. The push-out $T^*_{G_1} \sharp_{T^*_{G_0}} T^*_{G_2}$ with an embedded group $G_2 \cong D_{2l}$ (generated by G_0 and an element of order $2l$ commuting with an involution in G_0) yields the replacement of the mirror of order 2 by that of order $2l$; but, by a similar reasoning as above, l must be odd. Hence we get the tree in the left-hald side of Figure 11.

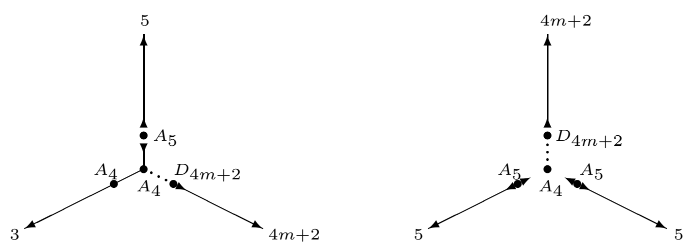


Figure 11

The right-hand side of Figure 11 with $m = 0$ is obtained from the tree in the left-hand side by the twist by the non-trivial element in $N(A_4)/(N(A_4) \bigcap N(A_5)) \cong Z_2$; the construction is similar to that of the third tree in Figure 8; note that $N(A_4) = S_4$. Further replacement gives the one with general m.

4.6. The following three trees of groups are simply obtained by the replacement method.

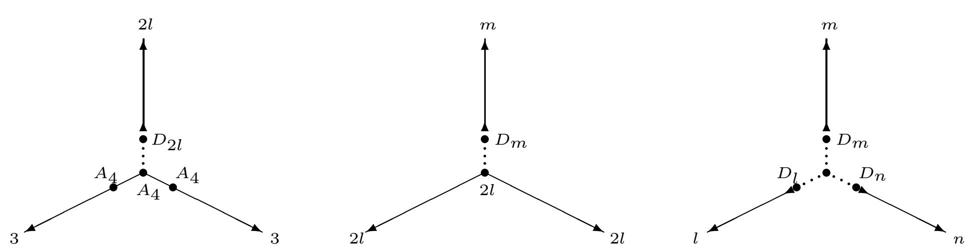


Figure 12

Here in the first tree in Figure 12, l must be odd; this follows from an argument similar to that below Figure 9. In the second one also, m is supposed to be odd; if one attempt to replace it by an even multiple of m, then the push-out gives a third tree. The third one is obtained by replacements of mirrors from $T^*_{D_2}$ (hence l, m, and n are even), and this is possible if and only if at least two of l, m, and n are not multiple of 4. The group attached to the certeral vertex is either D_2 if all of $l/2$, $m/2$, and $n/2$ is odd, or D_4 otherwise.

5. Proof of the theorem

In this section we prove that the triangle groups obtained in the previous section are the only possible ones, and complete the proof of the theorem. Let Γ be a p-adic Schwarzian triangle group of Mumford type. By Corollary 2.8, the tree T^*_Γ is a tripod. Let us fix a section $\iota_\Gamma\colon T^*_\Gamma \to \mathfrak{T}^*_\Gamma$, by which we decorate T^*_Γ to be a tree of groups. First we claim:

5.1. Lemma. *The tree of groups $(T^*_\Gamma, \Gamma_\bullet)$ is irreducible.*

Proof. Suffices to invoke Lemma 3.7, and the fact that there is no non-trivial covering over $\mathbb{P}^1_K$ which branches over at most 1 points. $\square$

Due to Proposition 3.9, the tree of groups T^*_Γ is isomorphic to the direct limit $\varinjlim T^*_{\Gamma_\bullet}$.

5.2. Lemma. *There exists a vertex $v \in \mathrm{Vert}(T^*_\Gamma)$ such that Γ_v is not a cyclic group.*

Proof. If not, all $T^*_{\Gamma_v}$ are straight lines, and all maps $T^*_{\Gamma_v \bigcap \Gamma_w} \to T^*_{\Gamma_v}$ are injective. Hence $\varinjlim T^*_{\Gamma_\bullet} \cong T^*_\Gamma$ must be a straight line. $\square$

In the sequel, we use the following notation: For $v_1, v_2 \in \mathrm{Vert}(T^*_\Gamma)$, we write $G_i = \Gamma_{v_i}$ $(i = 1, 2)$ and $G_0 = G_1 \bigcap G_2$. Let φ be the natural morphism

$$\varphi\colon T^*_{G_1} \sharp_{T^*_{G_0}} T^*_{G_2} \longrightarrow \varinjlim T^*_{\Gamma_\bullet} \cong T^*_\Gamma$$

of trees of groups.

5.3. First we claim that in $p > 5$ there is no triangle groups other than finite subgroups. We may assume G_1 is not a cyclic group.

Suppose G_0 is not cyclic. Then the map $\varrho^0_1\colon T^*_{G_0} \to T^*_{G_1}$ has the following properties (due to A.8):

(1) ϱ^0_1 maps the center of $T^*_{G_0}$ to that of $T^*_{G_1}$ (since the groups attached to outside the center are cyclic).
(2) The image of ϱ^0_1 is a union of half-lines starting at the center.

Hence, by virtue of Lemma 3.4, either $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$ must hold (note that in $T^*_{G_1}$ the center is fixed by G_1). This implies that the push-out $T^*_{G_1} \sharp_{T^*_{G_0}} T^*_{G_2}$ equals to either $T^*_{G_1}$ or to $T^*_{G_2}$.

Therefore, if Γ is not finite, then there must be v_1 and v_2 such that G_0 is cyclic; by virtue of Lemma 3.4, we may assume that G_2 is not cyclic. In this case, the

push-out $T^*_{G_1}\sharp_{T^*_{G_0}}T^*_{G_2}$ looks like as follows (cf. Example 3.2):

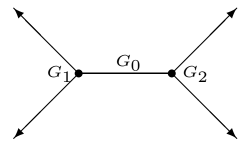


Figure 13

If there does not occur a push-out of form $T\sharp_{T^*_{G_0}}T^*_{G_3}$ in $\lim_{\longrightarrow} T^*_{\Gamma_\bullet}$, then clearly, it must have more than 3 ends. But even if it occurs, one can verify that the push-out $T\sharp_{T^*_{G_0}}T^*_{G_3}$ again has more than 3 ends; for example, if G_3 is not cyclic, and $\varrho^0_3\colon T^*_{G_0}\rightarrow T^*_{G_3}$ maps the straight line $T^*_{G_0}$ to a half-line (necessarily starting at the center), then the resulting push-out is still like as in Figure 13 with G_2 being replaced by G_3. Other cases are also verified similarly, even easier.

Hence, in any case, it is deduced that the tree T^*_Γ must have more than 3 ends, thereby the contradiction. Therefore, we have proved that in $p>5$ there is no infinite triangle groups.

5.4. Next we discuss the case $p=5$. In this case, only A_5 is exceptional. In other words, if there is no $v\in\mathrm{Vert}(T^*_\Gamma)$ with $G_v\cong A_5$, then the same argument as in the previous paragraph shows that Γ must be finite. Hence we may assume that $G_1\cong A_5$. Again, the argument similar to that in the previous paragraph shows that G_0 must not be cyclic. Then G_0 is isomorphic to either A_4, D_5, D_3, or D_2. If G_0 is not isomorphic to D_5, then the image of the map ϱ^0_1 contains a vertex fixed by the whole G_1 (cf. A.12), and again, Γ must be finite. Hence $G_0\cong D_5$. The possible isomorphism class of G_2 is A_5, D_{5l}. Now it is an easy combinatorics to check that the push-outs described in 4.1∼4.3 are the only possible ones.

The case $p=3$ is similar; in this case, A_4, S_4 and A_5 are exceptional. In each cases, the isomorphism class of G_0 is determined, and the easy observation shows that the trees appeared in 4.4 are the only possible ones.

Also in $p=2$, one can prove that the possible trees are among those in 4.5 and 4.6 basically by a similar idea; but the argument is more involved that the other cases. The possible non-trivial (isomorphism classes of) pairs (G_1,G_0) are as follows:

— (A_5,A_4), (A_5,D_2),
— (S_4,A_4), (S_4,D_4), (S_4,D_2),
— (A_4,D_2),
— (D_{2m+1},Z_2), (D_{2m},D_2).

Here, except for (D_{2m+1},Z_2), cyclic G_0 is discarded by the same reasoning as in 5.3; similarly, one sees that the following cases should be avoided:

$$\begin{aligned}(G_1,G_0,G_2) \;=\; & (A_5,D_2,A_5),(A_5,D_2,S_4),(S_4,D_2,S_4)\text{ (cf. Figure 9)},\\ & (A_4,D_2,A_4),(A_4,D_2,A_5),(A_4,D_2,S_4),(D_{2m+1},Z_2,D_{2n+1}).\end{aligned}$$

By a straightforward combinatorics (not very painful but tedious), one verifies that we have listed all the possible combinations in 4.5 and 4.6.

5.5. It remains to prove that the conjugacy class of a p-adic Schwarzian triangle group Γ of Mumford type is unique. As one finds in the description of T^*_Γ in the previous section, in each p, the abstract structure of the tree of groups T^*_Γ is determined by its index (e_0, e_1, e_∞); in particular, the abstract group structure of Γ is determined by its index. Since T^*_Γ is a tripod, i.e. has exactly three ends, and since giving three ends in Bruhat-Tits tree $\mathcal{T}_K$ determines a tripod, $\mathrm{PGL}(2, K)$ acts transitively on the set of admissible embeddings of T^*_Γ, and hence, on the set of embeddings of Γ in $\mathrm{PGL}(2, K)$. $\square$

A. Appendix: Trees of finite groups

A.1. This appendix is responsible for detailed description of the tree $\mathcal{T}^*_G$ and the tree of groups $(T^*_G, G_\bullet)$ for a finite subgroup $G \subset \mathrm{PGL}(2, K)$. First let us collect some facts on such subgroups, necessary for later use, which are either well-known (cf., for example, [Web99, Vol. II]) or are easy to verify:

(A.1.1) *Any finite subgroup $G \subset \mathrm{PGL}(2, K)$ is isomorphic to either a cyclic group (denoted by Z_m), a dihedral group (denoted by $D_m \cong Z_m \rtimes Z_2$), the tetrahedral group ($\cong A_4$), the octahedral group ($\cong S_4$), or to the icosahedral group ($\cong A_5$).*

(A.1.2) *Two isomoprhic finite subgroups are conjugate in* $\mathrm{PGL}(2, K)$.

(A.1.3) *Maximal cyclic subgroups of G of same order comprise a single conjugacy class in G except for the case $G \cong D_m$ with m even. If $G = \langle \theta, \varphi \mid \theta^m = \varphi^2 = (\theta\varphi)^2 = 1\rangle \cong D_m$ with m even, then $\langle\theta\rangle$, $\langle\varphi\rangle$, and $\langle\theta\varphi\rangle$ give the complete system of conjugacy classes of maximal cyclic subgroups.*

A.2. *Strategy of description.* Here is the general strategy for calculating $\mathcal{T}^*_G$:

(1) The tree $\mathcal{T}^*_G$ is the minimal one which contains all the mirrors of elements ($\neq 1$) of G, which are in bijection with maximal cyclic subgroups in G. We therefore first need to know how these mirrors are arranged in $\mathcal{T}_K$; the general principle for this will be given in Lemma A.4 below, by which we will see that the necessary data are cross-ratios of fixed points. Calculating these values is completely elementary, but needs a lot of computation; we will give the complete list of such data for $G \cong A_4$, S_4, and A_5 in the next appendix. We are thus able to describe the tree $\mathcal{T}^*_G$ perfectly.

(2) To describe $(T^*_G, G_\bullet)$, we further need to know the fixed locus in $\mathcal{T}_K$ of elliptic elements; the fixed locus contains the mirror, but they do not coincides in general, which has been already observed in Lemma 2.10.

A.3. Let $a = (a_0\colon a_1)$, $b = (b_0\colon b_1)$, $c = (c_0\colon c_1)$ and $d = (d_0\colon d_1)$ be four pairwise distinct K-rational points of $\mathbb{P}^1_K$. We are interested in the arrangement of two apartments $]a, b[$ and $]c, d[$ in $\mathcal{T}_K$. Let us define the cross-ratio

$$R(a, b; c, d) = \frac{(a_1c_0 - a_0c_1)(b_1d_0 - b_0d_1)}{(a_0b_1 - a_1b_0)(c_0d_1 - c_1d_0)}.$$

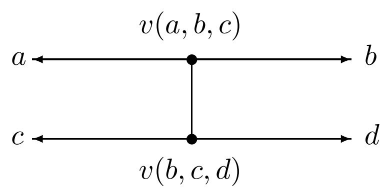
A.4. Lemma. *Let $\nu\colon K^\times \to \mathbb{Z}$ be the normalized (i.e., $\nu(\pi) = 1$) valuation.*

(1) If $|\nu(R(a, b; c, d))| = |\nu(R(b, a; c, d))| = 0$, then $]a, b[$ and $]c, d[$ intersects at exactly one vertex.

(2) If $|\nu(R(a,b;c,d))| = |\nu(R(b,a;c,d))| \neq 0$, then $]a,b[$ and $]c,d[$ are disjoint with the distance $|\nu(R(a,b;c,d))|$.

(3) If $|\nu(R(a,b;c,d))| \neq |\nu(R(b,a;c,d))|$, then the intersection of $]a,b[$ and $]c,d[$ is the path $[v(a,b,c), v(b,c,d)]$ of length $\max\{|\nu(R(a,b;c,d))|, |\nu(R(b,a;c,d))|\}$, where $v(z_0,z_1,z_2)$ for pairwise distinct three points $z_0,z_1,z_2 \in \mathbb{P}^1(K)$ is the unique vertex lying in the intersection of all $]z_i,z_j[$ for $i,j=0,1,2$, $i \neq j$.

Proof. First we recall how to calculate $v(z_0,z_1,z_2)$: Let Y_i ($i=0,1,2$) be a homogeneous coordinate of z_i, and choose $\alpha_i \in K^\times$ such that $\alpha_0 Y_0 + \alpha_1 Y_1 + \alpha_2 Y_2 = 0$. Then $v(z_0,z_1,z_2)$ is the similarity class of $\mathcal{O}_K \alpha_i Y_i + \mathcal{O}_K \alpha_j Y_j$ for any $i,j=0,1,2$, $i \neq j$. By this, it is easily checked that $d(v(a,b,c), v(b,c,d)) = |\nu(R(a,b;c,d))|$. Once it is checked, all the statements are clear, since the apartments $]a,b[$ and $]c,d[$ either do not intersect or intersect along a path (see Figure 14). $\square$



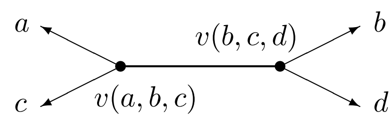


FIGURE 14: Arrangement of the apartments $]a,b[$ and $]c,d[$

A.5. *Folding.* For describing the quotient tree T^*_G, it is yet more convenient to have the following notion: Since any element $\gamma \neq 1$ has fixed points in $\Omega_G = \mathbb{P}^{1,\mathrm{an}}_K$, the image of the mirror $M(\gamma)$ is either a straight-line or a half-line; in the former case, $\varrho_G\colon M(\gamma) \to \varrho_G(M(\gamma))$ is an isomorphism, whereas in the latter, it is 2-to-1 except on one vertex. If this latter happens, we say that the mirror $M(\gamma)$ is *folded.*

(A.5.1) *The folding of the mirror $M(\gamma)$ occurs if and only if there exists $\theta \in G$ of order 2 which interchanges the two fixed points of γ, or equivalently, $\langle \gamma, \theta \rangle \cong D_m$, where m is the order of γ.*

Also, it is clear that the folding of mirrors depends only on the conjugacy classes; hence, in view of (A.1.3), one can easily tell which mirror is folded, by only checking existence or non-existence of dihedral subgroups of G. As a result, we get:

(A.5.2) *If $G \cong D_m$ with m even, $G \cong S_4$, or $G \cong A_5$, then all mirrors are folded. If $G \cong D_m$ with m odd (resp. A_4), only the mirror of elements of order m (resp. 2) is folded. If $G \cong Z_m$, no mirror is folded.*

A.6. *Cyclic case: $G \cong Z_n$.* For any residue characteristic p the tree $\mathcal{T}^*_G$ consists of only one apartment which is the mirror $M(\theta)$ of any element $\theta \neq 1$ in G. Since G acts on $\mathcal{T}^*_G$ trivially, the quotient tree T^*_G also consists of one straight-line whose ends corresponds to the two points above which $\mathbb{P}^1_K \to G\backslash\mathbb{P}^1_K$ ramifies.

A.7. *Convention.* In the following paragraphs, we only present the quotient tree T^*_G and the stabilizers in pictures for G a non-cyclic subgroups. One can check these by first drawing $\mathcal{T}^*_G$ by means of Lemma A.4 and the data in tables in the next

appendix; details are left to the reader (but, as a hint for the careful reader, we will exhibit in the end of the next appendix the picture of $\mathcal{T}_G^*$ for $G \cong A_5$ in $p = 2$). The pictures are subject to the following convention:

- Solid lines are the images of mirrors, while dotted segments are the ones which are not images of any mirror (recall that the tree $\mathcal{T}_G^*$ is not in general simply the union of mirrors).
- Ends are denoted by the arrow.
- If a mirror has the half-line as its image, then the starting point is denoted by the symbol ⤚, and the half-line starts at the vertex nearest it.
- The stabilizers of edges are omitted, since they are simply the intersection of the stabilizers of their end points. The number m placed by a vertex or an end indicates that the stabilizer is isomorphic to the cyclic group of order m.

Also, the unit length u (i.e. the distance of neighboring dots) are given in each picture.

A.8. *Generic cases.* In the cases $G \cong D_m$ (m: odd, $p \neq 2$) and $G \cong A_4$ ($p \neq 2, 3$), the trees T_G^* look like that in Figure 15; the unique dot signifies the vertex fixed by G, and the other parts are fixed by cyclic subgroups of orders in the index in Table 2. In the cases $G \cong D_m$ (m: even, $p \neq 2$), $G \cong S_4$ ($p \neq 2, 3$), and $G \cong A_5$ ($p \neq 2, 3, 5$), the trees T_G^* look like that in Figure 16 with the stabilizers subject to the similar rule.

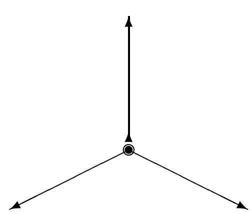


FIGURE 15

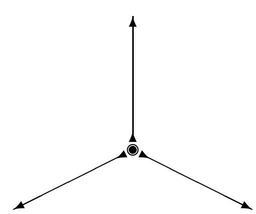


FIGURE 16

A.9. *Dihedral case:* $G \cong D_m$, $p = 2$.

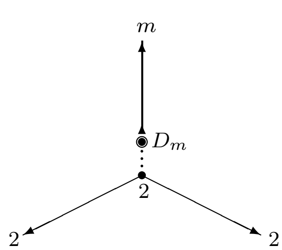


FIGURE 17: m: odd, $u = e$

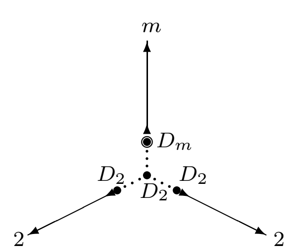


FIGURE 18: m: even, $u = \frac{e}{2}$

A.10. *Tetrahedral case:* $G \cong A_4$.

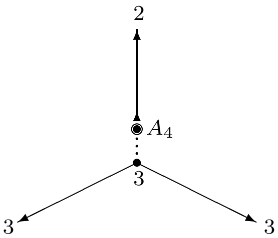


FIGURE 19: $p = 3$, $u = \frac{e}{2}$

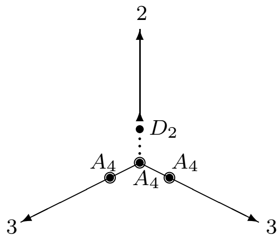


FIGURE 20: $p = 2$, $u = \frac{e}{2}$

A.11. *Octahedral case:* $G \cong S_4$.

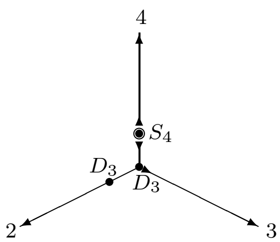


FIGURE 21: $p = 3$, $u = \frac{e}{2}$

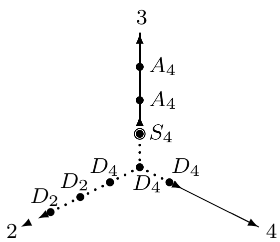


FIGURE 22: $p = 2$, $u = \frac{e}{4}$

A.12. *Icosahedral case:* $G \cong A_5$.

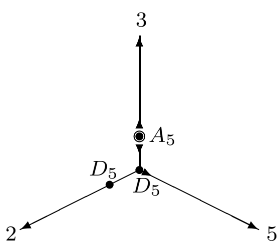


FIGURE 23: $p = 5$, $u = \frac{e}{4}$

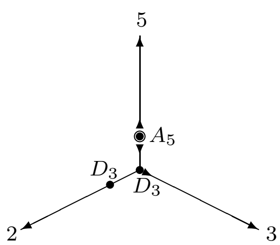


FIGURE 24: $p = 3$, $u = \frac{e}{2}$

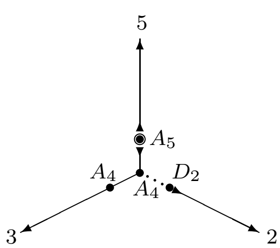


FIGURE 25: $p = 2$, $u = \frac{e}{2}$

B. Appendix: Combinatorial data

B.1. This appendix gives the tables of combinatorial data by which one can see how the mirrors of elements in finite subgroup in $\mathrm{PGL}(2, K)$ are arranged in $\mathcal{T}_K$ so that one can draw the picture of $\mathcal{T}_G^*$. The basic principle is as follows: Suppose G is a finite subgroup in $\mathrm{PGL}(2, K)$ and $\gamma, \theta \in G$ ($\gamma \neq 1,\ \theta \neq 1$). We assume that fixed points a, b (resp. c, d) of γ (resp. θ) are in $\mathbb{P}^1(K)$. Then the mirror of γ (resp. θ) is given by $]a, b[$ (resp. $]c, d[$). As we saw in Lemma A.4 the correlation between $M(g)$

and $M(h)$ can be calculated by the cross-ratios $R(a,b;c,d)$ and $R(b,a;c,d)$. If we have the complete list of these values for every pair of generators of maximal cyclic subgroups in G, we therefore can perfectly describe $\mathfrak{T}^*_G$.

B.2. *Convention.* For the subgroups $G \cong A_4, S_4, A_5$, which are given in the standard forms as presented below (this is allowed by (A.1.2)), we will give a complete list of maximal cyclic subgroups by choosing generators. To two of these generators, say γ and θ, we associate an expression $P(\gamma,\theta) = P(\theta,\gamma)$, which is either a number or an unordered pair of numbers. The meaning of $P(\gamma,\theta)$ is:

— If it is simply a number, then both $|\nu(R(a,b;c,d))|$ and $|\nu(R(b,a;c,d))|$ are equal to $|\nu(P(\gamma,\theta))|$.

— If it is a pair of numbers, say $P(\gamma,\theta) = \{s,t\}$, then, as a set, $\{|\nu(s)|, |\nu(t)|\}$ coincides with $\{|\nu(R(a,b;c,d))|, |\nu(R(b,a;c,d))|\}$.

B.3. *Case $G \cong A_4$ or S_4.* Since A_4 is a subgroup of S_4, the calculation can be mixed up. Let G be a subgroup isomorphic to S_4. By [Web99, §73] we may assume that G is generated by θ and χ with

$$\theta = \begin{bmatrix} \sqrt{i} & 0 \\ 0 & \frac{1}{\sqrt{i}} \end{bmatrix}, \qquad \chi = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ -\frac{1+i}{2} & \frac{1+i}{2} \end{bmatrix},$$

where i denotes a primitive 4-th root of unity. We set $\omega = \chi\theta\chi\theta^2$. The group G has three cyclic groups of order 4 generated by each of $\theta, \chi^2\theta^3, \chi\omega\theta^2$, four cyclic groups of order 3 generated by each of $\chi, \theta\chi\theta^3, \theta^2\chi\theta^2, \theta^3\chi\theta$, and six cyclic groups of order 2, not coming from those of order 4, generated by each of $\omega, \omega\theta^2, \omega\chi, \chi^2\theta, \omega\chi^2, \theta\chi^2$. Table 3 presents the $P(g,h)$ for any distinct two of these elements.

The group G has a subgroup isomorphic to A_4 generated by θ^2 and χ. It has four cyclic groups of order 3, $\langle\chi\rangle$, $\langle\theta^2\chi\theta^2\rangle$, $\langle\theta\chi\theta^3\rangle$, and $\langle\theta^3\chi\theta\rangle$, and three cyclic groups of order 2, $\langle\theta^2\rangle$, $\langle\chi\theta^2\chi^2\rangle$, and $\langle\chi^2\theta^2\chi\rangle$.

Table 3: $P(g,h)$ for g and h generators of cyclic subgroups in $G \cong S_4$. Here $\heartsuit = \{3,6\}$ and $\diamondsuit = \{4,12\}$.

	$\chi^2\theta^3$	$\chi\omega\theta^2$	χ	$\theta\chi\theta^3$	$\theta^2\chi\theta^2$	$\theta^3\chi\theta$	ω	$\omega\theta^2$	$\omega\chi$	$\chi^2\theta$	$\omega\chi^2$	$\theta\chi^2$
θ	2	2	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	2	2	$2\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$
$\chi\theta\chi^2$		2	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{2}$	$2\sqrt{2}$	2	2	$2\sqrt{2}$	$2\sqrt{2}$
$\chi^2\theta\chi$			$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$	2	2
χ				$\heartsuit$	$\heartsuit$	$\heartsuit$	2	$2\sqrt{3}$	2	$2\sqrt{3}$	2	$2\sqrt{3}$
$\theta\chi\theta^3$					$\heartsuit$	$\heartsuit$	$2\sqrt{3}$	2	$2\sqrt{3}$	2	2	$2\sqrt{3}$
$\theta^2\chi\theta^2$						$\heartsuit$	2	$2\sqrt{3}$	$2\sqrt{3}$	2	$2\sqrt{3}$	2
$\theta^3\chi\theta$							$2\sqrt{3}$	2	2	$2\sqrt{3}$	$2\sqrt{3}$	2
ω								2	$\diamondsuit$	$\diamondsuit$	$\diamondsuit$	$\diamondsuit$
$\omega\theta^2$									$\diamondsuit$	$\diamondsuit$	$\diamondsuit$	$\diamondsuit$
$\omega\chi$										2	$\diamondsuit$	$\diamondsuit$
$\chi^2\theta$											$\diamondsuit$	$\diamondsuit$
$\omega\chi^2$												2

B.4. *Case* $G \cong A_5$. By [loc. cit. §74] G is generated by θ and χ with

$$\theta = \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix}, \qquad \chi = \begin{bmatrix} \zeta+\zeta^4 & 1 \\ 1 & -(\zeta+\zeta^4) \end{bmatrix},$$

where ζ is a primitive 5-th root of unity. We set $\varphi = \chi\theta^{-1}\chi\theta\chi\theta^{-1}$, which sends $z \mapsto -1/z$. The group G has

— six cyclic subgroups of order 5: $\langle\theta\rangle$, $\langle\theta\chi\rangle$, $\langle\theta\chi\theta^3\rangle$, $\langle\theta^2\chi\theta^2\rangle$, $\langle\theta^3\chi\theta\rangle$, $\langle\theta^4\chi\rangle$,

— ten cyclic subgroups of order 3: $\langle\theta\chi\theta\rangle$, $\langle\theta^2\chi\rangle$, $\langle\theta^3\chi\rangle$, $\langle\theta^2\chi\theta\rangle$, $\langle\theta^3\chi\theta^4\rangle$, $\langle\theta\chi\varphi\rangle$, $\langle\theta^4\chi\varphi\rangle$, $\langle\theta^2\chi\varphi\theta^4\rangle$, $\langle\theta^2\chi\varphi\theta^2\rangle$, $\langle\theta^3\chi\varphi\theta\rangle$,

— and fifteen cyclic subgroups of order 2: $\langle\varphi\theta^i\rangle$, $\langle\theta^{-i}\chi\theta^i\rangle$, $\langle\theta^{-i}\chi\varphi\theta^i\rangle$ ($i = 0,1,2,3,4$).

Under these notation we have:

— $P(g,h) = \sqrt{5}$ if g and h are generators of cyclic groups of order 5 ($g \neq h$).

— $P(g,h) = \sqrt{3}\cdot 5^{1/4}$ if g (resp. h) is one of the generators of cyclic groups of order 5 (resp. 3).

All the rest are shown in the tables below:

Table 4: $P(g,h)$ for g and h generators of cyclic subgroups in $G \cong A_5$ of order 3. Here $\heartsuit = \{3,6\}$.

	$\theta^2\chi$	$\theta^3\chi$	$\theta^2\chi\theta$	$\theta^3\chi\theta^4$	$\theta\chi\varphi$	$\theta^4\chi\varphi$	$\theta^2\chi\varphi\theta^4$	$\theta^2\chi\varphi\theta^2$	$\theta^3\chi\varphi\theta$
$\theta\chi\theta$	$\heartsuit$	$\heartsuit$	3	3	$\heartsuit$	$\heartsuit$	$\heartsuit$	3	$\heartsuit$
$\theta^2\chi$		3	3	$\heartsuit$	$\heartsuit$	$\heartsuit$	$\heartsuit$	$\heartsuit$	3
$\theta^3\chi$			$\heartsuit$	3	$\heartsuit$	$\heartsuit$	3	$\heartsuit$	$\heartsuit$
$\theta^2\chi\theta$				$\heartsuit$	3	$\heartsuit$	$\heartsuit$	$\heartsuit$	$\heartsuit$
$\theta^3\chi\theta^4$					$\heartsuit$	3	$\heartsuit$	$\heartsuit$	$\heartsuit$
$\theta\chi\varphi$						3	3	$\heartsuit$	$\heartsuit$
$\theta^4\chi\varphi$							$\heartsuit$	$\heartsuit$	3
$\theta^2\chi\varphi\theta^4$								3	$\heartsuit$
$\theta^2\chi\varphi\theta^2$									3

Table 5: $P(g,h)$ for g (resp. h) a generator of cyclic group subgroups in $G \cong A_5$ of order 5 (resp. 2). Here $* = 2\cdot 5^{1/4}$.

	φ	$\varphi\theta$	$\varphi\theta^2$	$\varphi\theta^3$	$\varphi\theta^4$	χ	$\theta\chi\theta^4$	$\theta^2\chi\theta^3$	$\theta^3\chi\theta^2$	$\theta^4\chi\theta$	$\chi\varphi$	$\theta\chi\varphi\theta^4$	$\theta^2\chi\varphi\theta^3$	$\theta^3\chi\varphi\theta^2$	$\theta^4\chi\varphi\theta$
θ	2	2	2	2	2	*	*	*	*	*	*	*	*	*	*
$\theta\chi$	*	*	*	*	2	*	*	2	*	2	2	2	*	*	*
$\theta\chi\theta^3$	*	*	2	*	*	2	*	*	2	*	*	2	2	*	*
$\theta^2\chi\theta^2$	2	*	*	*	*	*	2	*	*	2	*	*	2	2	*
$\theta^3\chi\theta$	*	*	*	2	*	2	*	2	*	*	*	*	*	2	2
$\theta^4\chi$	*	2	*	*	*	*	2	*	2	*	2	*	*	*	2

Table 6: $P(g, h)$ for g (resp. h) a generator of cyclic group subgroups in $G \cong A_5$ of order 3 (resp. 2).

	φ	$\varphi\theta$	$\varphi\theta^2$	$\varphi\theta^3$	$\varphi\theta^4$	χ	$\theta\chi\theta^4$	$\theta^2\chi\theta^3$	$\theta^3\chi\theta^2$
$\theta\chi\theta$	2	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$
$\theta^2\chi$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	2	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	
$\theta^3\chi$	$2\sqrt{3}$	$\sqrt{6}$	2	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{3}$
$\theta^2\chi\theta$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	2	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$
$\theta^3\chi\theta^4$	$\sqrt{6}$	2	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$2\sqrt{3}$
$\theta\chi\varphi$	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{3}$	2	2	2	$\sqrt{6}$	$2\sqrt{3}$
$\theta^4\chi\varphi$	$2\sqrt{3}$	2	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	2	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$
$\theta^2\chi\varphi\theta^4$	$\sqrt{6}$	$2\sqrt{3}$	2	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	2	2	$\sqrt{6}$
$\theta^2\chi\varphi\theta^2$	2	$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	2	2
$\theta^3\chi\varphi\theta$	$\sqrt{6}$	$\sqrt{6}$	$2\sqrt{3}$	2	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	2

$\theta^4\chi\theta$	$\chi\varphi$	$\theta\chi\varphi\theta^4$	$\theta^2\chi\varphi\theta^3$	$\theta^3\chi\varphi\theta^2$	$\theta^4\chi\varphi\theta$	
$2\sqrt{3}$	$2\sqrt{3}$	2	$\sqrt{6}$	$\sqrt{6}$	2	$\theta\chi\theta$
$\sqrt{6}$	2	$\sqrt{6}$	2	$\sqrt{6}$	$\sqrt{6}$	$\theta^2\chi$
$2\sqrt{3}$	2	$\sqrt{6}$	$\sqrt{6}$	2	$2\sqrt{3}$	$\theta^3\chi$
$2\sqrt{3}$	$\sqrt{6}$	$\sqrt{6}$	2	$2\sqrt{3}$	2	$\theta^2\chi\theta$
$\sqrt{6}$	$\sqrt{6}$	2	$2\sqrt{3}$	2	$\sqrt{6}$	$\theta^3\chi\theta^4$
$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	$\theta\chi\varphi$
2	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$\theta^4\chi\varphi$
$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$\theta^2\chi\varphi\theta^4$
$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\sqrt{6}$	$\theta^2\chi\varphi\theta^2$
2	$\sqrt{6}$	$2\sqrt{3}$	$\sqrt{6}$	$2\sqrt{3}$	$2\sqrt{3}$	$\theta^3\chi\varphi\theta$

Table 7: $P(g, h)$ for g and h generators of cyclic group subgroups in $G \cong A_5$ of order 2. Here $\bigstar = \{4, 4\sqrt{5}\}$ and $\Diamond = \{4, 12\}$.

	$\varphi\theta$	$\varphi\theta^2$	$\varphi\theta^3$	$\varphi\theta^4$	χ	$\theta\chi\theta^4$	$\theta^2\chi\theta^3$	$\theta^3\chi\theta^2$
φ	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	2	$\bigstar$	$\Diamond$	$\Diamond$
$\varphi\theta$		$\bigstar$	$\bigstar$	$\bigstar$	$\Diamond$	$\bigstar$	2	$\bigstar$
$\varphi\theta^2$			$\bigstar$	$\bigstar$	$\bigstar$	$\Diamond$	$\Diamond$	$\bigstar$
$\varphi\theta^3$				$\bigstar$	$\bigstar$	2	$\bigstar$	$\Diamond$
$\varphi\theta^4$					$\Diamond$	$\Diamond$	$\bigstar$	2
χ						$\Diamond$	$\bigstar$	$\bigstar$
$\theta\chi\theta^4$							$\Diamond$	$\bigstar$
$\theta^2\chi\theta^3$								$\Diamond$
$\theta^3\chi\theta^2$								
$\theta^4\chi\theta$								
$\chi\varphi$								
$\theta\chi\varphi\theta^4$								
$\theta^2\chi\varphi\theta^3$								
$\theta^3\chi\varphi\theta^2$								

$\theta^4\chi\theta$	$\chi\varphi$	$\theta\chi\varphi\theta^4$	$\theta^2\chi\varphi\theta^3$	$\theta^3\chi\varphi\theta^2$	$\theta^4\chi\varphi\theta$	
$\bigstar$	2	$\diamondsuit$	$\bigstar$	$\bigstar$	$\diamondsuit$	φ
$\diamondsuit$	$\bigstar$	$\diamondsuit$	2	$\diamondsuit$	$\bigstar$	$\varphi\theta$
2	$\diamondsuit$	$\bigstar$	$\bigstar$	$\diamondsuit$	2	$\varphi\theta^2$
$\diamondsuit$	$\diamondsuit$	2	$\diamondsuit$	$\bigstar$	$\bigstar$	$\varphi\theta^3$
$\bigstar$	$\bigstar$	$\bigstar$	$\diamondsuit$	2	$\diamondsuit$	$\varphi\theta^4$
$\diamondsuit$	2	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	χ
$\bigstar$	$\bigstar$	2	$\bigstar$	$\bigstar$	$\bigstar$	$\theta\chi\theta^4$
$\bigstar$	$\bigstar$	$\bigstar$	2	$\bigstar$	$\bigstar$	$\theta^2\chi\theta^3$
$\diamondsuit$	$\bigstar$	$\bigstar$	$\bigstar$	2	$\bigstar$	$\theta^3\chi\theta^2$
	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	2	$\theta^4\chi\theta$
		$\bigstar$	$\diamondsuit$	$\diamondsuit$	$\bigstar$	$\chi\varphi$
			$\bigstar$	$\diamondsuit$	$\diamondsuit$	$\theta\chi\varphi\theta^4$
				$\bigstar$	$\diamondsuit$	$\theta^2\chi\varphi\theta^3$
					$\bigstar$	$\theta^3\chi\varphi\theta^2$

B.5. *Picture of $\mathcal{T}_G^*$ with $G \cong A_5$ and $p = 2$.* Mirrors are denoted by piecewise linear lines; both ends of them are indicated by arrows. Each generator of cyclic subgroups of order 3 is denoted twice at both ends of its mirror, while that of cyclic subgroups of order 2 is written once beside its mirror. The mirrors of generators of cyclic subgroups of order 5 are not drawn; they simply pass through the center of the picture. Every unit-segment is of length $e/2$.

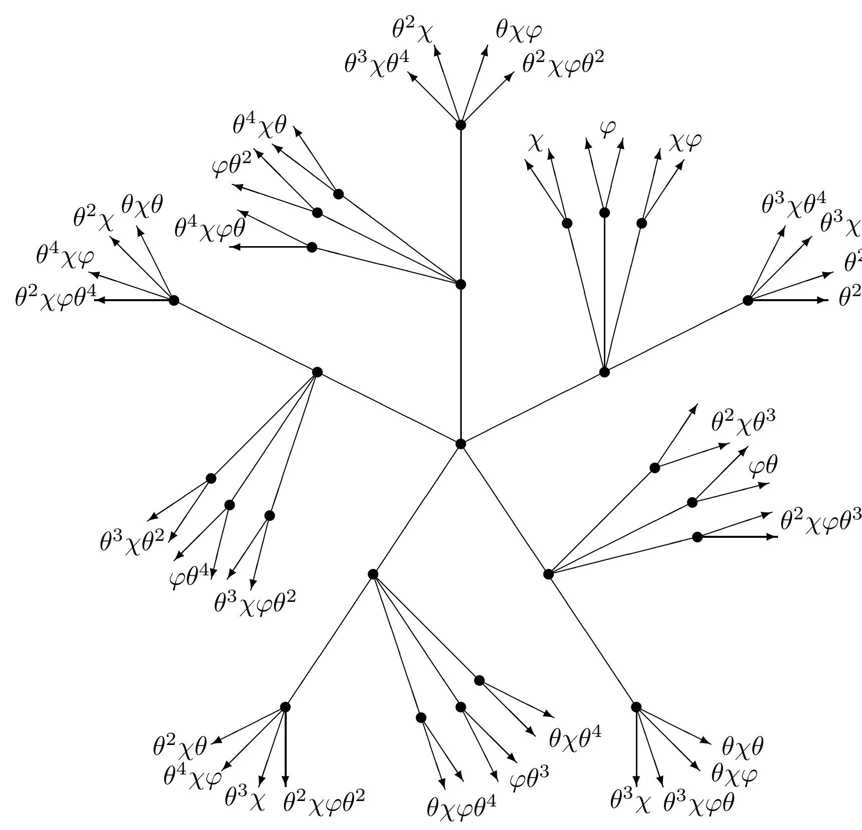


FIGURE 26: $\mathcal{T}_G^*$ for $G \cong A_5$ and $p = 2$

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